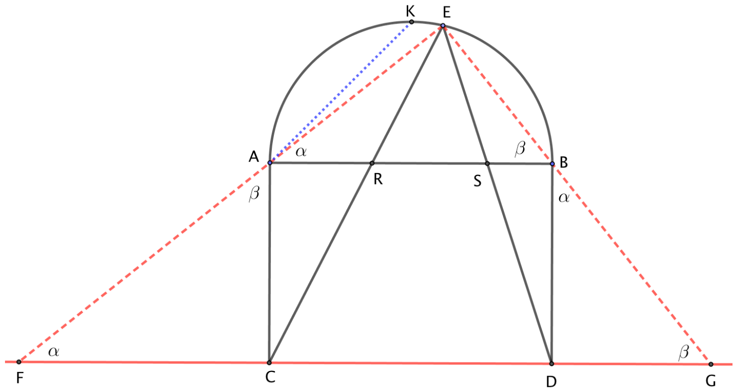
Galois club prize problem 2019A / A classical proof for a theorem of Fermat
by Xier Peng, March 2019



Introduction: In the diagram K is the midpoint of the semicircular arc AKB drawn upon the diameter AB. E is an arbitrary point on the arc. The line segments AC and BD are perpendicular against AB and have the same length[1] as AK. The lines EC and ED intersect AB at points R and S.

Fermat Theorem: $\mathrm{AB}^2 = \mathrm{AS}^2 + \mathrm{BR}^2$

A classical proof by similar triangles

First of all, we draw three auxiliary lines (shown in red): lines from E through A and B as well as a line connecting C and D. These lines intersect at points F and G.

We observe that the augmented diagram contains several pairs of similar triangles topping at the point E. We have $\Delta\mathrm{EAR}\sim\Delta\mathrm{EFC}$, $\Delta\mathrm{ERS}\sim\Delta\mathrm{ECD}$, $\Delta\mathrm{EAS}\sim\Delta\mathrm{EFD}$, $\Delta\mathrm{ERB}\sim\Delta\mathrm{ECG}$ and $\Delta\mathrm{EAB}\sim\Delta\mathrm{EFD}$, where Δ stands for "triangle" and $\sim$ indicates similarity. Why, for example, is $\Delta\mathrm{EAR}\sim\Delta\mathrm{EFC}$? Because all angles in these two triangles are pairwise equal. Clearly the same is true for the other pairs of triangles.

As we know, similar triangles have proportional sides. From the similarities noticed above we therefore obtain the following equations

$$\frac{\mathrm{ER}}{\mathrm{EC}} = \frac{\mathrm{ES}}{\mathrm{ED}} = \frac{\mathrm{AS}}{\mathrm{FD}} = \frac{\mathrm{BR}}{\mathrm{GC}} = \frac{\mathrm{AB}}{\mathrm{FG}} = k\,,$$

where k is the constant ratio of proportionality (in this case $0 < k < 1$). It follows that the left and right sides of our Theorem can be expressed as follows

$$\text{left side} = \mathrm{AB}^2 = (k\cdot\mathrm{FG})^2 = k^2\cdot\mathrm{FG}^2\,,$$

$$\text{right side} = \mathrm{AS}^2 + \mathrm{BR}^2 = (k\cdot\mathrm{FD})^2 + (k\cdot\mathrm{GC})^2 = k^2(\mathrm{FD}^2 + \mathrm{GC}^2)\,.$$

It remains to show that (*) $\mathrm{FG}^2 = \mathrm{FD}^2 + \mathrm{GC}^2$.The Theorem then follows by division through k^2.

To achieve that we notice one more pair of similar triangles namely $\Delta\mathrm{ACF}\sim\Delta\mathrm{GDB}$. Both are right triangles with two other angles α and β (Exercise: Show that they are equal in both triangles.) It follows that $\mathrm{FC}/\mathrm{AC} = \mathrm{BD}/\mathrm{DG} = \mathrm{AC}/\mathrm{DG}$ and hence $\mathrm{FC}\cdot\mathrm{DG} = \mathrm{AC}^2 = \mathrm{AB}^2/2 = \mathrm{CD}^2/2$ because $\mathrm{AC} = \mathrm{AK} = \mathrm{AB}/\sqrt{2} = \mathrm{CD}/\sqrt{2}$. So $\mathrm{CD}^2 = 2\mathrm{AC}^2 = 2\,\mathrm{FC}\cdot\mathrm{DG}$. Hence $\mathrm{GC}^2 = (\mathrm{CD}+\mathrm{DG})^2 = \mathrm{CD}^2 + 2\,\mathrm{CD}\cdot\mathrm{DG} + \mathrm{DG}^2 = 2\,\mathrm{FC}\cdot\mathrm{DG} + 2\,\mathrm{CD}\cdot\mathrm{DG} + \mathrm{DG}^2 = 2\,\mathrm{FD}\cdot\mathrm{DG} + \mathrm{DG}^2$.

Finally we have $\mathrm{FD}^2 + \mathrm{GC}^2 = \mathrm{FD}^2 + 2\,\mathrm{FD}\cdot\mathrm{DG} + \mathrm{DG}^2 = (\mathrm{FD}+\mathrm{DG})^2 = \mathrm{FG}^2$. **QED**

[1] Notice that $\mathrm{AK} = \mathrm{AB}/\sqrt{2}$ and hence $\mathrm{AK}^2 = \mathrm{AB}^2/2$.