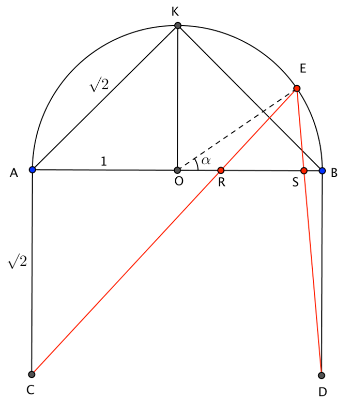
Galois club prize problem 2019A / A "modern" Cartesian proof for a theorem of Fermat
by Xier Peng, March 2019



Introduction: In the diagram K is the midpoint of the semicircular arc AKB drawn upon the diameter AB. E is an arbitrary point on the arc. The line segments AC and BD are perpendicular against AB and have the same length as AK. The lines EC and ED intersect AB at points R and S.

Fermat Theorem: $\mathrm{AB}^2 = \mathrm{AS}^2 + \mathrm{BR}^2$

A "modern" proof using Cartesian analytic geometry

We add coordinate axes on the diagram. We place origin at the point O, *x*-axis horizontally along the diameter AB and *y*-axis vertically along OK. We choose the unit length 1 to be equal with the length of the radius, that is OA = OB = 1. Then the length of AK is $\sqrt{2}$.

The position of the point E is given by the angle α between the lines OB and OE. The coordinates of E are then $x_{\mathrm{E}} = \cos\alpha$, $y_{\mathrm{E}} = \sin\alpha$. The coordinates of C are $x_{\mathrm{C}} = -1, y_{\mathrm{C}} = -\sqrt{2}$. The coordinates of D are $x_{\mathrm{D}} = 1, y_{\mathrm{D}} = -\sqrt{2}$.

Now we can write the equations for the lines CRE and DSE by using the familiar formula

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)\,,$$

which gives the equation of the line through points (x_1, y_1) and (x_2, y_2).

For the line CRE we put $x_1 = x_{\mathrm{C}} = -1, y_1 = y_{\mathrm{C}} = -\sqrt{2},\ x_2 = x_{\mathrm{E}} = \cos\alpha, y_2 = y_{\mathrm{E}} = \sin\alpha$, and get

$$(\mathrm{CRE}) \qquad y + \sqrt{2} = \frac{\sin\alpha + \sqrt{2}}{\cos\alpha + 1}(x + 1).$$

For DSE we get

$$(\mathrm{DSE}) \qquad y + \sqrt{2} = \frac{\sin\alpha + \sqrt{2}}{\cos\alpha - 1}(x - 1).$$

The *x*-coordinates of the points R and S are now obtained by putting $y = 0$ in the above equations and solving x. We get

$$x_{\mathrm{R}} = -1 + \sqrt{2}\,(\cos\alpha + 1)/\big(\sin\alpha + \sqrt{2}\big)\,, \quad x_{\mathrm{S}} = 1 + \sqrt{2}(\cos\alpha - 1)/\big(\sin\alpha + \sqrt{2}\big).$$

Now it is a routine (though a bit tedious) task to compute the lengths of AS and BR and calculate the value $\mathrm{AS}^2 + \mathrm{BR}^2$ which turns out to be 4 which indeed is precisely AB^2. **QED**